BRX-TH-473
BOW-PH-117
HUTP-00/A018
hep-th/0006141

M-Theory tested by $\mathcal{N} = 2$ Seiberg-Witten Theory [1]

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Abstract

Methods are reviewed for computing the instanton expansion of the prepotential for $\mathcal{N} = 2$ Seiberg-Witten (SW) theory with *non*-hyperelliptic curves. These results, when compared with the instanton expansion obtained from the microscopic Lagrangian, provide detailed tests of M-theory.

Group theoretic regularities of $\mathcal{F}_{1-\mathrm{inst}}$ allow one to "reverse engineer" a SW curve for SU(N) with two antisymmetric representations and $N_f \leq 3$ fundamental hypermultiplet representations, a result not yet available by other methods. Consistency with M-theory requires a curve of infinite order.

[1]Based on a talk by H.J. Schnitzer at the *Workshop on Strings, Duality and Geometry,* University of Montreal, March 2000.

[2] Research supported by the DOE under grant DE–FG02–92ER40706.
[3] Research supported in part by the National Science Foundation under grant no. PHY94-07194 through the ITP Scholars Program.

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[5]Research supported in part by the DOE under grant DE–FG02–92ER40706.

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1. Objectives

We present a method for obtaining precise tests of M-theory using $\mathcal{N}=2$ Seiberg-Witten (SW) supersymmetric (susy) gauge theory [1]. Although one believes in M-theory, it must nevertheless be subjected to detailed verification for the same reasons that one subjects quantum electrodynamics to precision tests. In our context, M-theory provides SW curves for low-energy effective $\mathcal{N}=2$ susy gauge theories, which in principle allows one to compute the instanton expansion of the prepotential of the theory. The results of this calculation must be compared with calculations of the instanton contributions to the prepotential from the microscopic Lagrangian. It is this comparison which provides the tests we are concerned with. It should be noted that what we are considering is the ability of M-theory to make detailed non-perturbative predictions for field theory, as we consider the limit in which gravity has decoupled.

M-theory provides SW curves for effective $\mathcal{N}=2$ susy gauge theories with hypermultiplets in both the fundamental representation [2, 3] and in higher representations [4]. Since the (hyperelliptic) curves from the former were initially obtained from purely field-theoretic considerations, we regard these as *post*-dictions of M-theory, though the agreement is gratifying. In order to obtain genuine tests of M-theory we need to consider situations for which it is not known how to obtain SW curves from field-theoretic arguments alone; for example, $\mathcal{N}=2$ SU(N) gauge theory with a hypermultiplet in the symmetric or antisymmetric representation [4]. In such cases M-theory gives the only known predictions of the relevant SW curves, which happen to be *non*-hyperelliptic curves. If one can extract the instanton expansion for these examples, and compare these to results from a microscopic calculation, one will have genuine tests of M-theory. The problem we faced is that there were no known methods to obtain the instanton expansion. Our solution to this issue will be our main theme.

2. Seiberg-Witten Theory

Consider $\mathcal{N}=2$ susy Yang-Mills theory in $d=4$ dimensions, with gauge group $\mathcal{G}$, together with hypermultiplets in some representation R. This theory can be described by a microscopic Lagrangian

$$\mathcal{L}_{\rm micro} = \frac{1}{4g^2}\,F^a_{\mu\nu}\,F^{\mu\nu a} \,+\, \frac{\theta}{32\pi^2}\,F^a_{\mu\nu}\,\tilde{F}^{\mu\nu a} \,+\, D_\mu\phi^+ D^\mu\phi \,+\, {\rm tr}[\phi,\phi^+]^2 \\ +\ \text{fermion} + \text{hypermultiplet terms}, \tag{2.1}$$

with $\mu,\nu = 1$ to 4 and $a = 1$ to dim $\mathcal{G}$. The field strength $F_{\mu\nu}$ and the scalar field ϕ belong to

the adjoint representation. The vacuum is described by the condition

$$[\phi, \phi^+] \,=\, 0. \tag{2.2}$$

Rotate ϕ^a to the Cartan subalgebra, in which case

$$\mathrm{diag}\,(\phi) \,=\, (a_1, a_2, \ldots)\,, \qquad \text{with } \sum_i a_i = 0. \tag{2.3}$$

If all the a_i are distinct, this generically breaks $\mathcal{G}$ to $\mathrm{U}(1)^{\mathrm{rank}\,\mathcal{G}}$. If only ϕ acquires a vacuum expectation value (vev), we define this as the Coulomb branch, which is our focus.

Seiberg and Witten [1] formulated the exact solution of the low-energy description of $\mathcal{N} = 2$ susy gauge theories in terms of an effective (Wilsonian) action accurate to two derivatives of the fields, *i.e.*

$$\mathcal{L}_{\mathrm{eff}} \,=\, \frac{1}{4\pi}\mathrm{Im}\left(\int \mathrm{d}^4\theta \frac{\partial \mathcal{F}(A)}{\partial A_i}\bar{A}_i + \frac{1}{2}\int \mathrm{d}^2\theta \frac{\partial^2 \mathcal{F}(A)}{\partial A_i\,\partial A_j} W_i^{\alpha}\, W_{\alpha,j}\right) \,+\, \text{higher derivatives}, \tag{2.4}$$

where A^i are $\mathcal{N} = 1$ chiral superfields ($i = 1$ to $\mathrm{rank}\,\mathcal{G}$), $\mathcal{F}(A)$ is the holomorphic prepotential, and W^i is the gauge field strength. In components the effective action is

$$\begin{aligned}\mathcal{L}_{\mathrm{eff}} \,=\, & \frac{1}{4}\mathrm{Im}(\tau_{ij})\, F^i_{\mu\nu} F^{\mu\nu j} \,+\, \frac{1}{4}\mathrm{Re}(\tau_{ij})\, F^i_{\mu\nu}\tilde{F}^{\mu\nu\, j} \\ & + \partial_\mu (a^+)^j \partial^\mu (a_D)_j \,+\, \text{fermions}.\end{aligned} \tag{2.5}$$

We define the order parameters a_i as in (2.3), $(a_D)_j = \frac{\partial \mathcal{F}(a)}{\partial a_j}$ are the dual order parameters, and

$$\tau_{ij} = \frac{\partial^2 \mathcal{F}(a)}{\partial a_i \partial a_j}, \tag{2.6}$$

is the coupling matrix. Note that $\mathrm{Im}(\tau_{ij}) \geq 0$ for positive kinetic energies.

The holomorphic prepotential can be expressed in terms of a perturbative part and infinite series of instanton contributions as

$$\mathcal{F}(A) \,=\, \mathcal{F}_{\mathrm{classical}}(A) \,+\, \mathcal{F}_{1-\mathrm{loop}}(A) \,+\, \sum_{d=1}^{\infty} \Lambda^{(2N - I(R))d} \mathcal{F}_{\mathrm{d-inst}}(A), \tag{2.7}$$

where we have specialized the instanton terms to $\mathrm{SU}(N)$ as an illustration. Due to a non-renormalization theorem, the perturbative expansion for (2.7) terminates at 1-loop, though there is an infinite series of non-perturbative instanton contributions. In (2.7), Λ is the quantum scale (Wilson cutoff) and $I(R)$ is the Dynkin index of matter hypermultiplet(s) of representation R.

The Seiberg-Witten data which (in principle) allow one to reconstruct the prepotential are:

1) A suitable Riemann surface or algebraic curve, dependent on moduli u_i, or equivalently on the order parameters a_i.
2) A preferred meromorphic 1-form $\lambda \equiv$ SW differential.
3) A canonical basis of homology cycles on the surface (A_k, B_k).
4) Computation of period integrals

$$2\pi i a_k = \oint_{A_k} \lambda, \qquad\qquad 2\pi i a_{D,k} = \oint_{B_k} \lambda, \tag{2.8}$$

where recall $a_{D,k} = \frac{\partial \mathcal{F}(a)}{\partial a_k}$ is the dual order parameter. The program is:

i) find the Riemann surface or algebraic curve appropriate to the given matter content,
ii) compute the period integrals,
iii) integrate these to find $\mathcal{F}(a)$, and
iv) test against results from $\mathcal{L}_{\rm micro}$ when possible.

What classes of SW curves are encountered for simple, classical groups $(\mathrm{SU}, \mathrm{SO}, \mathrm{Sp})$ with matter hypermultiplets consistent with asymptotic freedom?

a) hyperelliptic curves

$$y^2 \, + \, 2A(x)y \, + \, B(x) \, = \, 0, \tag{2.9}$$

for pure gauge theory + matter hypermultiplets in the fundamental representation.

b) cubic non-hyperelliptic curves

$$y^3 \, + \, 2A(x)y^2 \, + \, B(x)y \, + \, \epsilon(x) \, = \, 0, \tag{2.10}$$

which occurs for

i) $\mathrm{SU}(N)$ + 1 antisymmetric $+(N_f < N+2)$ fundamental hypermultiplets.
ii) $\mathrm{SU}(N)$ + 1 symmetric $+(N_f < N-2)$ fundamental hypermultiplets.

c) curves of infinite order from

i) elliptic models, or
ii) decompactifications of elliptic models.

Example: $\mathrm{SU}(N)+$ 2 antisymmetric and $N_f \leq 4$ fundamental hypermultiplets.

The main task in extracting instanton predictions from curves such as (2.10) is the computation of the period integrals (2.8), and the integration of $\partial\mathcal{F}(a)/\partial a_k$ to obtain $\mathcal{F}(a)$. There are two principal (complementary) methods to evaluate the period integrals for *hyperelliptic* curves. These are Picard-Fuchs differential equations for the period integrals [13], and direct evaluation of the period integrals by asymptotic expansion [14, 15, 16].

The problem we face is how to evaluate period integrals

$$\oint \lambda = \oint \frac{xdy}{y}, \tag{2.11}$$

for non-hyperelliptic curves such as (2.10). For the cubic curve, the exact solution is too complicated to be useable, while for curves of higher order, even exact solutions are not possible. Numerical solutions are of no interest, as we want to study the analytic behavior of $\mathcal{F}(a)$ on the order parameters.

3. M-theory and the Riemann Surface

The seminal work on this subject is by Witten [3], who considers IIA string theory lifted to M-theory. It is convenient to use the language of IIA theory in describing the brane structure. Consider SU(N) gauge theory with either an antisymmetric or symmetric matter hypermultiplet [4]. The M-theory picture is

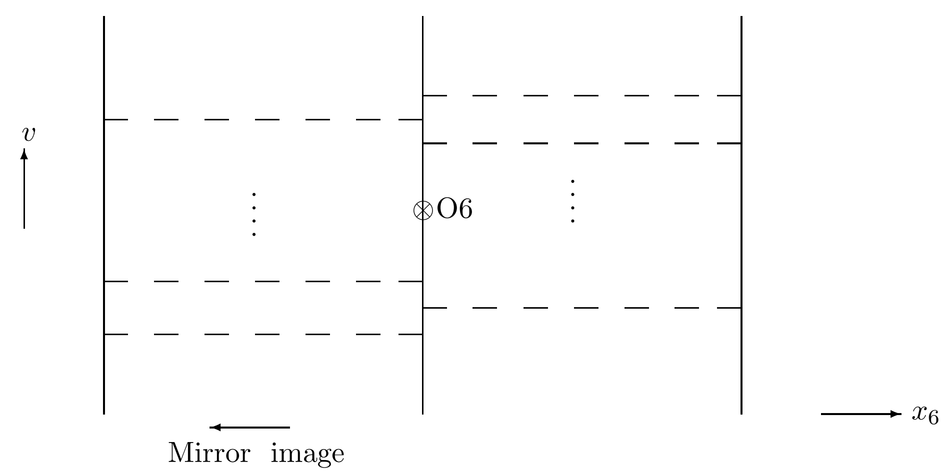

Figure 1

There are 3 parallel NS 5-branes with N D4-branes suspended between each, and an O6-plane on the central NS 5-brane. In the absence of the orientifold, one would have SU(N) $\times$ SU(N) with matter in the $(N, \bar{N}) \oplus (\bar{N}, N)$ representation. The orientifold "identifies" the two SU(N) factors, projecting to the diagonal subgroup, giving a single SU(N) factor with one hypermultiplet in the antisymmetric representation for O6$^-$, or one hypermultiplet in the symmetric representation for O6$^+$ [4]. It is important to note that the orientifold induces a $\mathbb{Z}_2$ involution in the curve. The NS 5-branes are at $x_7 = x_8 = x_9 = 0$, with *classically* fixed values of x_6. The D4-branes have world-volume x_0, x_1, x_2, x_3, x_6, with ends at fixed values of x_6, which gives a *macroscopic* world-volume on the D4-brane as $d = 4$. The M-theory picture gives rise to Riemann surface, which is the SW curve for this situation.

4. Hyperelliptic Perturbation Theory

We have developed a systematic scheme for the instanton expansion for prepotentials associated to non-hyperelliptic curves [5]–[11], which will be illustrated for the case of SU(N) gauge theory with one hypermultiplet in the antisymmetric representation [5]. The curve is given by (2.10) where $L^2 = \Lambda^{N+2}$, with Λ the quantum scale of the theory, and

$$\begin{aligned} \epsilon = L^6 \ ; \qquad 2A(x) &= \left[f(x) + 3\,L^2\right] , \\ f(x) = x^2 \prod_{i=1}^{N} (x - e_i) \ ; \qquad B(x) &= L^2 \left[f(-x) + 3\,L^2\right] , \\ \text{involution :} \qquad y \to \frac{L^4}{y}, &\qquad x \to -x. \end{aligned} \tag{4.1}$$

It is fruitful to regard the last term $\epsilon = L^6$ in (2.10) as a perturbation. The intuition is that this involves a much higher power of the quantum scale in (2.10) than the other terms, and geometrically it means separating the right-most 5-brane far from the remaining two NS 5-branes.

To zeroth approximation we consider (2.10) with $\epsilon = 0$, which is then a hyperelliptic curve, and can be analyzed by previously available methods [14]–[16]. This approximation gives $\mathcal{F}_{1-\mathrm{loop}}$ correctly, but it is not adequate for $\mathcal{F}_{1-\mathrm{inst}}$, so one needs to go beyond the hyperelliptic approximation. We present a systematic expansion in ϵ, which is not the same as an expansion in L, as the coefficient functions $A(x)$ and $B(x)$ depend on L. The perturbative expansion in ϵ for the solution is

$$y_i = \bar{y}_i + \delta y_i = \bar{y}_i + \alpha_i \epsilon + \beta_i \epsilon^2 + \cdots , \qquad i = 1, 2, 3, \tag{4.2}$$

which for our example gives to first order,

$$\begin{aligned} y_1(x) &= -A - r - \frac{L^6 (A - r)}{2Br} + \cdots \\ y_2(x) &= -A + r + \frac{L^6 (A + r)}{2Br} + \cdots \\ y_3(x) &= -\frac{L^6}{B} + \cdots \end{aligned} \tag{4.3}$$

with subscripts denoting the appropriate sheet, and with $r = \sqrt{A^2 - B}$. It is straightforward to go to higher orders in ϵ.

We note that sheet 3 is disconnected in any finite order of our perturbation expansion, so we need only consider y_1 and y_2. We need the SW differential for these two sheets, with the SW

differential for sheet 1

$$\lambda_1 = x\frac{dy_1}{y_1}, \tag{4.4}$$

and λ_2 obtained from (4.4) by $r \to -r$. Information on sheet 3 is obtained from the involution symmetry in (4.1). The expansion (4.2) induces a comparable expansion for λ, whereby

$$\begin{aligned} \lambda_1 &= (\lambda_1)_I + (\lambda_1)_{II} + \cdots , & (4.5)\\ \lambda_I &= \frac{x\left(\frac{A'}{A} - \frac{B'}{2B}\right)}{\sqrt{1-\frac{B}{A^2}}}\, dx, \\ \lambda_{II} &= -\frac{L^6\left(A - \frac{B}{2A}\right)}{B^2\sqrt{1-\frac{B}{A^2}}}\, dx, & (4.6) \end{aligned}$$

up to terms that do not contribute to period integrals. Notice that λ_I is the SW differential obtained from the hyperelliptic approximation ($\epsilon = 0$) to (2.10), and completely determines $\mathcal{F}_{1-\text{loop}}$ and a part of the 1-instanton term, while $\lambda_{II} \sim \mathcal{O}(L^2)$, so is of 1-instanton order. Further terms contribute only to 2-instanton order and higher, so (4.6) is all that is needed to 1-instanton order.

In order to express the solutions to our problem with economical notation, we define "*residue functions*", $R_k(x)$, $S(x)$, $S_0(x)$, and $S_k(x)$, where

$$\frac{R_k(x)}{(x-e_k)} = \frac{3}{f(x)}, \tag{4.7}$$

and

$$S(x) = \frac{f(-x)}{f^2(x)} = \frac{S_0(x)}{x^2} = \frac{S_k(x)}{(x-e_k)^2}. \tag{4.8}$$

The functions $S(x)$ and $S_k(x)$ play a crucial role for understanding the general features of the instanton expansion of SW problems.

The branch-cuts are centered on the e_i and connect sheets y_1 and y_2 as shown in Fig. 2

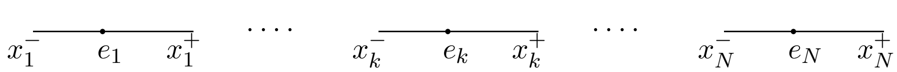

Figure 2

The order parameters are computed in a canonical homology basis. For the order parameters a_k we have the basis A_k, as shown in Fig. 3,

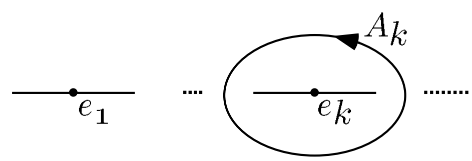

Figure 3

and the basis B_k for the dual order parameters $a_{D,k}$ as shown in Fig. 4,

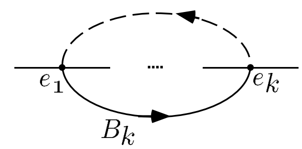

Figure 4

The cycle B_k connects sheets y_1 and y_2, with the solid line on sheet y_1 and the dashed line on y_2, with B_k passing through the branch-cut as shown. To compute (2.8), one only needs $(\lambda_1 - \lambda_2)$, so that we only need terms odd under $r \to -r$. The order parameter is

$$\begin{aligned} 2\pi i\, a_k &= \oint_{A_k} \lambda \simeq \oint_{A_k} (\lambda_I + \lambda_{II} + \cdots) \\ &= \oint_{A_k} dx \left[\frac{x\left(\frac{A'}{A} - \frac{B'}{2B}\right)}{\sqrt{1-\frac{B}{A^2}}} - L^6 \frac{\left(A - \frac{B}{2A}\right)}{B^2\sqrt{1-\frac{B}{A^2}}} \right]. \end{aligned} \tag{4.9}$$

The second term does not contribute to $\mathcal{O}(L^2)$, as there are no poles at $x = e_k$. Thus to this order

$$a_k = e_k + L^2 \left[\frac{\partial S_k}{\partial x}(e_k) - R_k(e_k) \right] + \cdots \tag{4.10}$$

The computation of the dual order parameter is considerably more complicated, with the result

$$\begin{aligned} 2\pi i a_{D,k} &= 2\pi i (a_{D,k})_I + 2\pi i (a_{D,k})_{II} \\ &= 2\pi i \frac{\partial}{\partial a_k}[\mathcal{F}_{\text{classical}} + \mathcal{F}_{1-\text{loop}}] + L^2 \frac{\partial}{\partial a_k}[-2S_0(0) + \sum_{k=1}^{N} S_k(a_k)], \end{aligned} \tag{4.11}$$

so that the one-instanton prepotential for SU(N) gauge theory with one massless antisymmetric hypermultiplet [5] turns out to be

$$\mathcal{F}_{1-\text{inst}} = \frac{1}{2\pi i} \left[-2\, S_0(0) + \sum_k S_k(a_k) \right], \tag{4.12}$$

which is a prediction of M-theory that may be tested against microscopic calculations. This is presently possible for SU(N) with $N \leq 4$, since SU(2) + $\begin{smallmatrix}\square\\\square\end{smallmatrix}$ = SU(2) (pure gauge theory); SU(3) + $\begin{smallmatrix}\square\\\square\end{smallmatrix}$ = SU(3) + $\square$; and SU(4) + $\begin{smallmatrix}\square\\\square\end{smallmatrix}$ = SO(6) + $\square$. In each of these three cases, (4.12) agrees with 1-instanton calculations from $\mathcal{L}_{\rm micro}$ [17]. For $N \geq 5$, (4.12) should be regarded as predictions of M-theory, awaiting testing. The fact that (4.12) agrees with microscopic calculations, when available, after a long derivation, with distinct methods, is already impressive.

There are further applications of hyperelliptic perturbation theory [5]–[11], where the analysis is very similar to that sketched above.

One may add hypermultiplets in the fundamental representation, and hypermultiplets with non-zero masses. For SU(N) gauge theory with an antisymmetric representation and $N_f < N+2$, which is described by a cubic SW curve [4], one finds [7]:

$$2\pi i \mathcal{F}_{1-{\rm inst}} = \sum_{k=1}^{N} S_k(a_k) - 2S_m(-\tfrac{1}{2}m), \tag{4.13}$$

where

$$S_k(a_k) = \frac{(-1)^N \prod_{j=1}^{N_f}(a_k + M_j) \prod_{i=1}^{N}(a_k + a_i + m)}{(a_k + \frac{1}{2}m)^2 \prod_{i\neq k}^{N}(a_k - a_i)^2}, \tag{4.14}$$

$$S_m(-\tfrac{1}{2}m) = \frac{(-1)^N \prod_{j=1}^{N_f}(M_j - \frac{1}{2}m)}{\prod_{k=1}^{N}(a_k + \frac{1}{2}m)}, \tag{4.15}$$

where M_j (m) is the mass of the hypermultiplet in the fundamental (resp. antisymmetric) representation. Eqs. (4.14) and (4.15) agree with scaling limits taking M_j and/or $m \to \infty$. Eqs. (4.13)–(4.15) provide additional tests of M-theory, since SU(2) + $\begin{smallmatrix}\square\\\square\end{smallmatrix}$ + N_f $\square$ = SU(2) + N_f $\square$, SU(3) + $\begin{smallmatrix}\square\\\square\end{smallmatrix}$ + N_f $\square$ = SU(3) + (N_f + 1) $\square$, both of which agree with microscopic instanton calculations [17].

An additional SU(N) theory with a cubic SW curve is SU(N) + $\square\!\square$ + N_f $\square$, with $N_f < N+2$. Here the result is [7], [8]:

$$2\pi i \mathcal{F}_{1-{\rm inst}} = \sum_{k=1}^{N} S_k(a_k), \tag{4.16}$$

$$S_k(a_k) = \frac{(-1)^N (a_k + \frac{1}{2}m)^2 \prod_{j=1}^{N_f}(a_k + M_j) \prod_{i=1}^{N}(a_k + a_i + m)}{\prod_{i\neq k}^{N}(a_k - a_i)^2}. \tag{4.17}$$

This agrees with the microscopic calculation of Slater [12].

Whenever the predictions of the cubic SW curves obtained from M-theory have been tested, agreement has been found with those of microscopic calculations. However, there remain numerous further opportunities to subject M-theory predictions to testing.

5. Universality

If one examines the cases discussed in the previous section, one observes certain universal features:

(i) The natural variables for this class of problems are the order parameters $\{a_k\}$ and not the gauge invariant moduli.

(ii) The 1-instanton contribution to the prepotential can be written as [6, 7, 14]

$$2\pi i \mathcal{F}_{1-\mathrm{inst}} = \sum_{k=1}^{N} S_k(a_k), \tag{5.1}$$

for SU(N) $+$ N_f $\Box$ or SU(N) $+$ $\Box\!\Box$ $+$ N_f $\Box$, and

$$2\pi i \mathcal{F}_{1-\mathrm{inst}} = \sum_{k=1}^{N} S_k(a_k) - 2S_m(-\tfrac{1}{2}m), \tag{5.2}$$

for SU(N) $+$ $\begin{smallmatrix}\Box\\ \Box\end{smallmatrix}$ $+$ N_f $\Box$ [5, 7].

Define $S(x)$ which generalizes (4.8) as

$$S(x) \;=\; \frac{S_k(x)}{(x-a_k)^2} \;=\; \frac{S_m(x)}{(x+\frac{1}{2}m)^2}. \tag{5.3}$$

We tabulate the known $S(x)$ for SU(N) in the first three entries of Table 1, where we include all generic cases of asymptotically free $\mathcal{N}=2$ SU(N) gauge theories. A careful examination of the first three rows of Table 1 leads to the following empirical rules. $S(x)$ is given as the product of the following factors, each corresponding to a different $\mathcal{N}=2$ multiplet in a given representation of SU(N):

(1) Pure gauge multiplet factor

$$\frac{1}{\prod_{i=1}^{N}(x-a_i)^2}. \tag{5.4}$$

(2) Fundamental representation $\Box$. A factor

$$(x+M_j) \tag{5.5}$$

for each hypermultiplet of mass M_j in the fundamental representation.

(3) Symmetric representation $\square\!\square$. A factor

$$(-1)^N\,(x+\tfrac{1}{2}m)^2\,\prod_{i=1}^{N}(x+a_i+m) \tag{5.6}$$

for each hypermultiplet of mass m in the symmetric representation.

(4) Antisymmetric representation $\begin{smallmatrix}\square\\\square\end{smallmatrix}$. A factor

$$\frac{(-1)^N}{(x+\frac{1}{2}m)^2}\,\prod_{i=1}^{N}(x+a_i+m) \tag{5.7}$$

for each hypermultiplet of mass m in the antisymmetric representation.

From these empirical rules, we predict $S(x)$ for the last two entries of Table 1. There are similar empirical rules for SO and Sp [10].

Hypermultiplet Representations	$S(x)$
SU(N) + N_f fund. (M_j) ($N_f \le 2N$) (ref. [14])	$\frac{\prod_{j=1}^{N_f}(x+M_j)}{\prod_{i=1}^{N}(x-a_i)^2}$
SU(N) + 1 sym. (m) + N_f fund. (M_j) ($N_f \le N-2$) (ref. [6, 7])	$\frac{(-1)^N(x+\frac{1}{2}m)^2\prod_{i=1}^{N}(x+a_i+m)\prod_{j=1}^{N_f}(x+M_j)}{\prod_{i=1}^{N}(x-a_i)^2}$
SU(N) + 1 anti.(m) + N_f fund. (M_j) ($N_f \le N+2$) (ref. [5, 7])	$\frac{(-1)^N\prod_{i=1}^{N}(x+a_i+m)\prod_{j=1}^{N_f}(x+M_j)}{(x+\frac{1}{2}m)^2\prod_{i=1}^{N}(x-a_i)^2}$
SU(N) + 2 anti. (m_1, m_2) + N_f fund. (M_j) ($N_f \le 4$) (ref. [9])	$\frac{\prod_{i=1}^{N}(x+a_i+m_1)\prod_{i=1}^{N}(x+a_i+m_2)\prod_{j=1}^{N_f}(x+M_j)}{(x+\frac{1}{2}m_1)^2(x+\frac{1}{2}m_2)^2\prod_{i=1}^{N}(x-a_i)^2}$
SU(N) + 1 anti. (m_1) + 1 sym. (m_2)	$\frac{(x+\frac{1}{2}m_2)^2\prod_{i=1}^{N}(x+a_i+m_1)\prod_{i=1}^{N}(x+a_i+m_2)}{(x+\frac{1}{2}m_1)^2\prod_{i=1}^{N}(x-a_i)^2}$

Table 1: The function $S(x)$ for SU(N) gauge theory, with different matter content.

Thus from these regularities we predict [9] for SU(N) + 2 $\begin{smallmatrix}\square\\\square\end{smallmatrix}$ $+N_f\square$ with $N_f \le 4$:

$$2\pi i\mathcal{F}_{1-\mathrm{inst}} = \sum_{k=1}^{N} S_k(a_k) - 2S_{m_1}(-\tfrac{1}{2}m_1) - 2S_{m_2}(-\tfrac{1}{2}m_2), \tag{5.8}$$

where $S_k(a_k)$ and $S_m(-\frac{1}{2}m)$ are constructed from the 4^{th} entry of Table 1, even though no SW curve is available from M-theory!

The predictions of Table 1 and (5.8) can be tested as follows:

1) $\mathrm{SU}(2) + 2\,\begin{smallmatrix}\square\\\square\end{smallmatrix} + N_f\square = \mathrm{SU}(2) + N_f\square, (N_f \leq 3)$.

2) $\mathrm{SU}(3) + 2\,\begin{smallmatrix}\square\\\square\end{smallmatrix} + N_f\square = \mathrm{SU}(3) + (N_f+2)\square, (N_f \leq 3)$.

3) Limit m_1 or $m_2 \to \infty$ reduces to $\mathrm{SU}(N) + \begin{smallmatrix}\square\\\square\end{smallmatrix} + N_f\square$.

In each of these three tests, our predicted $\mathcal{F}_{1-\mathrm{inst}}$ finds agreement. However, it should be emphasized that there is no known derivation of the empirical rules of (5.4)-(5.7). This is a problem that deserves consideration from first principles.

6. Reverse Engineering a Curve

Although there is no known SW curve for $\mathrm{SU}(N)$ gauge theory with two antisymmetric representations and $N_f \leq 3$ hypermultiplets, one can attempt to reverse engineer a curve from the information in Table 1 and (5.8). The strategy is

1) $\mathcal{F}_{\mathrm{classical}} + \mathcal{F}_{1-\mathrm{loop}}$ from perturbation theory.

2) $\mathcal{F}_{1-\mathrm{inst}}$ as predicted in Table 1 and (5.8).

3) These two steps imply that $a_{D,k} = \frac{\partial \mathcal{F}}{\partial a_k}$ is known to 1-instanton accuracy.

4) Reproduce this expression from period integrals of a Riemann surface, to be constructed from the above data.

5) Ensure that the proposed Riemann surface is consistent with M-theory.

Since Witten has shown [3] that for $\mathrm{SU}(N) \times \mathrm{SU}(N) \times \overset{m \text{ factors}}{\cdots} \times \mathrm{SU}(N)$ the corresponding curve is

$$y^{m+1} + \cdots = 0,$$

which results from $m+1$ parallel NS 5-branes, and N D4-branes suspended between neighboring pairs of NS 5-branes. However, for $(\mathrm{SU}(N))^m$, with $m \geq 3$, we have shown [8] that to attain 1-instanton accuracy, one *only* needs a quartic approximation

$$y^4 + \cdots = 0,$$

to the full y^{m+1} curve. Therefore, we only need a quartic curve to reproduce the prepotential to 1-instanton accuracy. The most general quartic curve consistent with M-theory is of the form [3, 9]

$$\begin{aligned} &L^4\, j_1(x)\, P_2(x)\, t^2 \;+\; L\, P_1(x)\, t \;+\; P_0(x) \;+\; L\, j_0(x)\, P_{-1}(x)\, \frac{1}{t} \\ &+ L^4\, {j_0}^2(x)\, j_{-1}(x)\; P_{-2}(x)\, \frac{1}{t^2} = 0, \end{aligned} \tag{6.1}$$

where $j_n(x)$ are associated to the N_f flavors in the fundamental representation, and $P_n(x)$ are associated to the positions of D4-branes.

We argue that (6.1) is incomplete if consistency with M-theory is demanded, with the result of a curve of infinite order, and therefore an infinite chain of NS 5-branes and orientifolds. To see the origin of this assertion, recall that the brane picture for SU(N) gauge theory with an antisymmetric representation of mass m is shown in Fig. 1, and repeated in Fig. 5, showing only the NS 5-branes and the O6$^-$ plane for clarity.

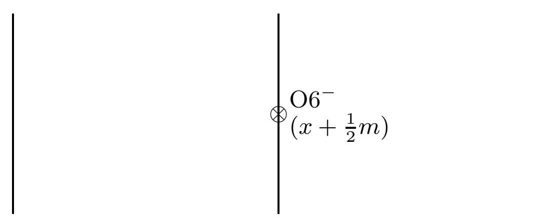

Figure 5

Therefore, for SU(N) gauge theory with two hypermultiplets in the antisymmetric representation of masses m_1 and m_2, we expect *at least* the brane structure in Fig. 6.

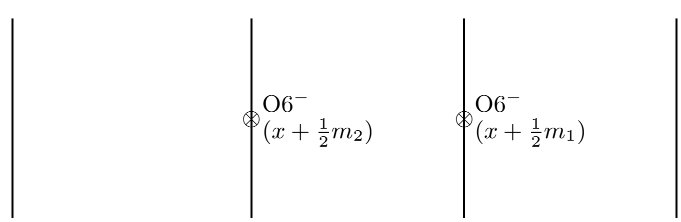

Figure 6

Again in Fig. 6, only the NS 5-branes and O6$^-$ are shown, while the N D4-branes connecting the NS 5-branes and the flavor D6-branes are not shown for clarity. The first observation is that to satisfy all possible mirrors, one must have an infinite chain of NS 5-branes and O6$^-$ orientifolds, since one must satisfy the reflections in *each* of the O6$^-$ orientifold planes separately. A portion of this chain is shown in Fig. 7, which differs from Fig. 6 in that the positions of D4-branes and D6-(flavor) branes are shown. One can check that all the necessary mirrors about any given O6$^-$ orientifold plane are satisfied.

Observe that if $m_2 \to \infty$, most of the D4-branes, D6-branes and O6$^-$ planes slide off to infin-

ity, leaving us with the configuration of Fig. 5 for SU(N) and an antisymmetric representation of mass m_1. Thus the infinite chain of NS 5-branes and O6$^-$ orientifolds, a portion of which is shown in Fig. 7, yields a curve of infinite order. The construction of such curves is described in the talk of S.G. Naculich at this workshop.

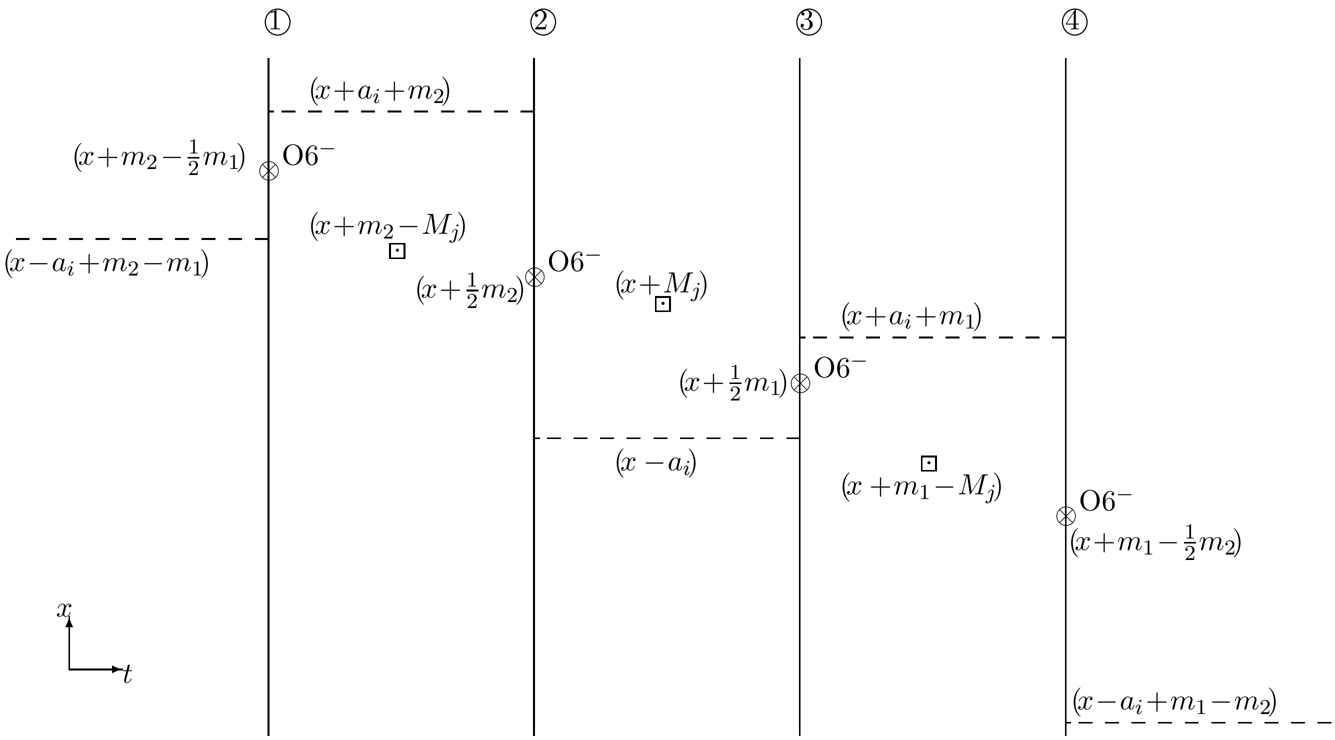

Figure 7:

1) *vertical lines* |: parallel, equally spaced NS 5-branes.

2) *dashed lines* −−: N parallel D4-branes connect pairs of adjacent NS 5-branes.

3) $\otimes$: O6$^-$ orientifold planes.

4) $\boxdot$: D6-(flavor) branes.

Due to mirrors, the picture must extend infinitely to right and left.

9. Concluding Remarks

There are a number of open problems which should be addressed. An incomplete list is:

1) Compute $\mathcal{F}_{1-\mathrm{inst}}$ from $\mathcal{L}_{\mathrm{micro}}$ for all the cases described in Table 1, so as to extend the test of M-theory. In every case where a test can be made, agreement has been found.

2) Explain *group-theoretically* the entries for $S(x)$ in Table 1, and the rules abstracted from these tables.

3) Extend the predictions of non-hyperelliptic curves to regions of moduli space for which the hyperelliptic perturbation theory is not valid.

4) Enlarge the connections to integrable models.

As we have discussed, $\mathcal{N} = 2$ SW theory presents many varied opportunities for testing M-theory predictions for sypersymmetric gauge theories. These deserve to be explored further to increase our confidence in M-theory.

Acknowledgement

We wish to thank the organizers of the workshop E. D'Hoker, D.H. Phong, and S.T. Yau for the opportunity to present the results of our group in such a stimulating atmosphere.

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